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Research Article

Strong Convergence Theorems for Infinitely Nonexpansive Mappings in Hilbert Space

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We introduce a modified Ishikawa iterative process for approximating a fixed point of two infinitely nonexpansive self-mappings by using the hybrid method in a Hilbert space and prove that the modified Ishikawa iterative sequence converges strongly to a common fixed point of two infinitely nonexpansive self-mappings.

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1. Introduction

Let C be a nonempty closed convex subset of a Hilbert space H , T a self-mapping of C . Recall that T is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$.

Construction of fixed points of nonexpansive mappings via Mann's iteration [1] has extensively been investigated in literature (see, e.g., [2–5] and reference therein). But the convergence about Mann's iteration and Ishikawa's iteration is in general not strong (see the counterexample in [6]). In order to get strong convergence, one must modify them. In 2003, Nakajo and Takahashi [7] proposed such a modification for a nonexpansive mapping T .

Consider the algorithm,

$$\begin{aligned}x_0 &\in C \text{ chosen arbitrarily,} \\y_n &= \alpha_n x_n + (1 - \alpha_n)Tx_n, \\C_n &= \{v \in C : \|y_n - v\| \leq \|x_n - v\|\}, \\Q_n &= \{v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0\}, \\x_{n+1} &= P_{C_n \cap Q_n}(x_0),\end{aligned}\tag{1.1}$$

where P_C denotes the metric projection from H onto a closed convex subset C of H . They prove the sequence $\{x_n\}$ generated by that algorithm (1.1) converges strongly to a fixed point of T provided that the control sequence $\{\alpha_n\}$ is chosen so that $\sup_{n \geq 0} \alpha_n < 1$.

Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings of C , $\{\lambda_n\}_{n=1}^\infty$ a sequence of nonnegative numbers in $[0, 1]$. For each $n \geq 1$, defined a mapping W_n of C into itself as follows:

$$\begin{aligned}
 U_{n,n+1} &= I, \\
 U_{n,n} &= \lambda_n T_n U_{n,n+1} + (1 - \lambda_n) I, \\
 U_{n,n-1} &= \lambda_{n-1} T_{n-1} U_{n,n} + (1 - \lambda_{n-1}) I, \\
 &\vdots \\
 U_{n,k} &= \lambda_k T_k U_{n,k+1} + (1 - \lambda_k) I, \\
 U_{n,k-1} &= \lambda_{k-1} T_{k-1} U_{n,k} + (1 - \lambda_{k-1}) I, \\
 &\vdots \\
 U_{n,2} &= \lambda_2 T_2 U_{n,3} + (1 - \lambda_2) I, \\
 W_n &= U_{n,1} = \lambda_1 T_1 U_{n,2} + (1 - \lambda_1) I.
 \end{aligned} \tag{1.2}$$

Such a mapping W_n is called the W -mapping generated by T_n, T_{n-1}, \dots, T_1 and $\lambda_n, \lambda_{n-1}, \dots, \lambda_1$; see [8].

In this paper, motivated by [9], for any given $x_i \in C$ ($i = 0, 1, \dots, q, q \in \mathbb{N}$ is a fixed number), we will propose the following iterative progress for two infinitely nonexpansive mappings $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$ in a Hilbert space H :

$$\begin{aligned}
 x_0, x_1, \dots, x_q &\in C \text{ chosen arbitrarily,} \\
 y_n &= \alpha_n x_n + (1 - \alpha_n) W_n^{(1)} z_{n-q}, \\
 z_n &= \bar{\alpha}_n x_n + (1 - \bar{\alpha}_n) W_n^{(2)} x_n, \\
 C_n &= \left\{ v \in K : \|y_n - v\|^2 \leq \|x_n - v\|^2 + (1 - \alpha_n) \left(\|x_{n-q} - x^*\|^2 - \|x_n - x^*\|^2 \right) \right\}, \\
 Q_n &= \{ v \in K : \langle x_n - v, x_n - x_q \rangle \leq 0 \}, \\
 x_{n+1} &= P_{C_n \cap Q_n}(x_q), n \geq q
 \end{aligned} \tag{1.3}$$

and prove, $\{x_n\}$ converges strongly to a fixed point of $\{T_n^{(1)}\}$ and $\{T_n^{(2)}\}$.

We will use the notation:

\rightharpoonup for weak convergence and \rightarrow for strong convergence.

$\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω -limit set of x_n .

2. Preliminaries

In this paper, we need some facts and tools which are listed as lemmas below.

Lemma 2.1 (see [10]). *Let H be a Hilbert space, C a nonempty closed convex subset of H , and T a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. If $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I - T)x_n\}$ converges strongly to y , then $(I - T)x = y$.*

Lemma 2.2 (see [11]). *Let C be a nonempty bounded closed convex subset of a Hilbert space H . Given also a real number $a \in \mathbb{R}$ and $x, y, z \in H$. Then the set $D := \{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v \rangle + a\}$ is closed and convex.*

Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C , where C is a nonempty closed convex subset of a strictly convex Banach space E . Given a sequence $\{\lambda_n\}_{n=1}^\infty$ in $[0, 1]$, one defines a sequence $\{W_n\}_{n=1}^\infty$ of self-mappings on C by (1.2). Then one has the following results.

Lemma 2.3 (see [8]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E , $\{T_n\}_{n=1}^\infty$ a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, for every $x \in C$ and $k \geq 1$ the limit $\lim_{n \rightarrow \infty} U_{n,k}x$ exists.*

Remark 2.4. It can be known from Lemma 2.3 that if D is a nonempty bounded subset of C , then for $\varepsilon > 0$ there exists $n_0 \geq k$ such that $\sup_{x \in D} \|U_{n,k}x - U_kx\| \leq \varepsilon$ for all $n > n_0$.

Remark 2.5. Using Lemma 2.3, we can define a mapping $W : C \rightarrow C$ as follows:

$$Wx = \lim_{n \rightarrow \infty} W_nx = \lim_{n \rightarrow \infty} U_{n,1}x \quad (2.1)$$

for all $x \in C$. Such a W is called the W -mapping generated by T_1, T_2, \dots and $\lambda_1, \lambda_2, \dots$. Since W_n is nonexpansive mapping, $W : C \rightarrow C$ is also nonexpansive. Indeed, observe that for each $x, y \in C$,

$$\|Wx - Wy\| = \lim_{n \rightarrow \infty} \|W_nx - W_ny\| \leq \|x - y\|. \quad (2.2)$$

If $\{x_n\}$ is a bounded sequence in C , then we put $D = \{x_n : n \geq 0\}$. Hence, it is clear from Remark 2.4 that for $\varepsilon > 0$ there exists $N_0 \geq 1$ such that for all $n > N_0$, $\|W_nx_n - Wx_n\| = \|U_{n,1}x_n - U_1x_n\| \leq \sup_{x \in D} \|U_{n,1}x - U_1x\| \leq \varepsilon$. This implies that

$$\lim_{n \rightarrow \infty} \|W_nx_n - Wx_n\| = 0. \quad (2.3)$$

Lemma 2.6 (see [8]). *Let C be a nonempty closed convex subset of a strictly convex Banach space E . Let $\{T_n\}_{n=1}^\infty$ be a sequence of nonexpansive self-mappings on C such that $\bigcap_{n=1}^\infty F(T_n) \neq \emptyset$ and let $\{\lambda_n\}$ be a sequence in $(0, b]$ for some $b \in (0, 1)$. Then, $F(W) = \bigcap_{n=1}^\infty F(T_n)$.*

3. Strong Convergence Theorem

Theorem 3.1. *Let C be a closed convex subset of a Hilbert space H and let $\{W_n^{(1)}\}$ and $\{W_n^{(2)}\}$ be defined as (1.2). Assume that $\alpha_n \leq a$ for all n and for some $0 < a < 1$, and $\{\bar{\alpha}_n\} \in [b, c]$ for all n and $0 < b < c < 1$. If $F = \bigcap_{n=1}^{\infty} [F(T_n^{(1)}) \cap F(T_n^{(2)})] \neq \emptyset$, then $\{x_n\}$ generated by (1.3) converges strongly to $P_F(x_q)$.*

Proof. Firstly, we observe that C_n is convex by Lemma 2.2. Next, we show that $F \subset C_n$ for all n .

Indeed, for all $x^* \in F$,

$$\begin{aligned}
 \|y_n - x^*\|^2 &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|z_{n-q} - x^*\|^2 \\
 &= \|x_n - x^*\|^2 + (1 - \alpha_n) (\|z_{n-q} - x^*\|^2 - \|x_n - x^*\|^2), \\
 \|z_{n-q} - x^*\|^2 &= \|\bar{\alpha}_{n-q} x_{n-q} + (1 - \bar{\alpha}_{n-q}) W_{n-q}^{(2)} x_{n-q} - x^*\|^2 \\
 &= \bar{\alpha}_{n-q} \|x_{n-q} - x^*\|^2 + (1 - \bar{\alpha}_{n-q}) \|W_{n-q}^{(2)} x_{n-q} - x^*\|^2 \\
 &\quad - \bar{\alpha}_{n-q} (1 - \bar{\alpha}_{n-q}) \|W_{n-q}^{(2)} x_{n-q} - x_{n-q}\|^2 \\
 &\leq \bar{\alpha}_{n-q} \|x_{n-q} - x^*\|^2 + (1 - \bar{\alpha}_{n-q}) \|x_{n-q} - x^*\|^2 \\
 &\quad - \bar{\alpha}_{n-q} (1 - \bar{\alpha}_{n-q}) \|W_{n-q}^{(2)} x_{n-q} - x_{n-q}\|^2 \\
 &= \|x_{n-q} - x^*\|^2 - \bar{\alpha}_{n-q} (1 - \bar{\alpha}_{n-q}) \|W_{n-q}^{(2)} x_{n-q} - x_{n-q}\|^2 \\
 &\leq \|x_{n-q} - x^*\|^2.
 \end{aligned} \tag{3.1}$$

Therefore,

$$\|y_n - x^*\|^2 \leq \|x_n - x^*\|^2 + (1 - \alpha_n) (\|x_{n-q} - x^*\|^2 - \|x_n - x^*\|^2). \tag{3.2}$$

That is $x^* \in C_n$ for all $n \geq q$. Next we show that $F \subset Q_n$ for all $n \geq q$.

We prove this by induction. For $n = q$, we have $F \subset C = Q_q$. Assume that $F \subset Q_n$ for all $n \geq q + 1$, since x_{n+1} is the projection of x_q onto $C_n \cap Q_n$, so

$$\langle x_{n+1} - z, x_q - x_{n+1} \rangle \geq 0, \quad \forall z \in C_n \cap Q_n. \tag{3.3}$$

As $F \subset C_n \cap Q_n$ by the induction assumption, the last inequality holds, in particular, for all $x^* \in F$. This together with definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Hence $F \subset C_n \cap Q_n$ for all $n \geq q$.

Notice that the definition of Q_n implies $x_n = P_{Q_n} x_q$. This together with the fact $F \subset Q_n$ further implies $\|x_n - x_q\| \leq \|x^* - x_q\|$ for all $x^* \in F$.

The fact $x_{n+1} \in Q_n$ asserts that $\langle x_{n+1} - x_n, x_n - x_q \rangle \geq 0$ implies

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_q) - (x_n - x_q)\|^2 \\ &= \|x_{n+1} - x_q\|^2 - \|x_n - x_q\|^2 - 2\langle x_{n+1} - x_n, x_n - x_q \rangle \\ &\leq \|x_{n+1} - x_q\|^2 - \|x_n - x_q\|^2 \longrightarrow 0 \quad (n \longrightarrow \infty). \end{aligned} \quad (3.4)$$

We now claim that $\|W^{(1)}x_n - x_n\| \rightarrow 0$ and $\|W^{(2)}x_n - x_n\| \rightarrow 0$. Indeed,

$$\begin{aligned} \|x_n - W_n^{(1)}z_{n-q}\| &= \frac{\|x_n - y_n\|}{1 - \alpha_n} \\ &\leq \frac{\|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|}{1 - \alpha_n}, \end{aligned} \quad (3.5)$$

since $x_{n+1} \in C_n$, we have

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + (1 - \alpha_n)(\|x_{n-q} - x^*\|^2 - \|x_n - x^*\|^2) \longrightarrow 0. \quad (3.6)$$

Thus

$$\|x_n - W_n^{(1)}z_{n-q}\| \longrightarrow 0. \quad (3.7)$$

We now show $\lim_{n \rightarrow \infty} \|W_n^{(2)}x_n - x_n\| = 0$. Let $\{\|W_{n_k}^{(2)}x_{n_k} - x_{n_k}\|\}$ be any subsequence of $\{\|W_n^{(2)}x_n - x_n\|\}$. Since C is a bounded subset of H , there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that

$$\lim_{j \rightarrow \infty} \|x_{n_{k_j}} - x^*\| = \limsup_{k \rightarrow \infty} \|x_{n_k} - x^*\| := r. \quad (3.8)$$

Since

$$\begin{aligned} \|x_{n_{k_j}} - x^*\| &\leq \|x_{n_{k_j}} - W_{n_{k_j}}^{(1)}z_{n_{k_j}-q}\| + \|W_{n_{k_j}}^{(1)}z_{n_{k_j}-q} - x^*\| \\ &\leq \|x_{n_{k_j}} - W_{n_{k_j}}^{(1)}z_{n_{k_j}-q}\| + \|z_{n_{k_j}-q} - x^*\|, \end{aligned} \quad (3.9)$$

it follows that $r = \lim_{j \rightarrow \infty} \|x_{n_{k_j}} - x^*\| \leq \liminf_{j \rightarrow \infty} \|z_{n_{k_j}} - x^*\|$. By (3.1), we have

$$\|z_{n_{k_j}} - x^*\| \leq \|x_{n_{k_j}} - x^*\|^2. \quad (3.10)$$

Hence

$$\limsup_{j \rightarrow \infty} \|z_{n_{k_j}} - x^*\| \leq \lim_{j \rightarrow \infty} \|x_{n_{k_j}} - x^*\| = r. \quad (3.11)$$

Thus,

$$\lim_{j \rightarrow \infty} \|z_{n_{k_j}} - x^*\| = r = \lim_{j \rightarrow \infty} \|x_{n_{k_j}} - x^*\|. \quad (3.12)$$

Using (3.1) again, we obtain that

$$\bar{\alpha}_{n_{k_j}-q} \left(1 - \bar{\alpha}_{n_{k_j}-q}\right) \|W_{n_{k_j}-q}^{(2)} x_{n_{k_j}-q} - x_{n_{k_j}-q}\|^2 \leq \|x_{n_{k_j}-q} - x^*\|^2 - \|z_{n_{k_j}-q} - x^*\|^2 \rightarrow 0. \quad (3.13)$$

This imply that $\lim_{j \rightarrow \infty} \|W_{n_{k_j}}^{(2)} x_{n_{k_j}} - x_{n_{k_j}}\| = 0$. For the arbitrariness of $\{x_{n_k}\} \subset \{x_n\}$, we have $\lim_{n \rightarrow \infty} \|W_n^{(2)} x_n - x_n\| = 0$ and

$$\|z_n - x_n\| = (1 - \bar{\alpha}_n) \|W_n^{(2)} x_n - x_n\| \rightarrow 0. \quad (3.14)$$

Thus, by (3.4), (3.7) and (3.14), we have

$$\begin{aligned} \|W_n^{(1)} x_n - x_n\| &\leq \|W_n^{(1)} x_n - W_n^{(1)} z_{n-q}\| + \|W_n^{(1)} z_{n-q} - x_n\| \\ &\leq \|z_{n-q} - x_n\| + \|W_n^{(1)} z_{n-q} - x_n\| \\ &\leq \|W_n^{(1)} z_{n-q} - x_n\| + \|z_{n-q} - x_{n-q}\| + \|x_{n-q} - x_{n-q+1}\| \\ &\quad + \|x_{n-q+1} - x_{n-q+2}\| + \cdots + \|x_{n-1} - x_n\| \\ &\rightarrow 0. \end{aligned} \quad (3.15)$$

Since $\lim_{n \rightarrow \infty} \|W_n^{(1)} x_n - W^{(1)} x_n\| = 0$ and $\lim_{n \rightarrow \infty} \|W_n^{(2)} x_n - W^{(2)} x_n\| = 0$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \|W^{(1)} x_n - x_n\| &= 0, \\ \lim_{n \rightarrow \infty} \|W^{(2)} x_n - x_n\| &= 0. \end{aligned} \quad (3.16)$$

Thus, using (3.16), Lemma 2.1, and the boundedness of $\{x_n\}$, we get that $\emptyset \neq \omega_w(x_n) \subset F$. Since $x_n = P_{Q_n}(x_q)$ and $F \subset Q_n$, we have $\|x_n - x_q\| \leq \|x^* - x_q\|$ where $x^* := P_F(x_q)$. By the weak lower semicontinuity of the norm, we have $\|w - x_q\| \leq \|x^* - x_q\|$ for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F$, we must have $w = x^*$ for all $w \in \omega_w(x_n)$. Hence $x_n \rightarrow x^* = P_F(x_q)$ and

$$\begin{aligned} \|x_n - x^*\|^2 &= \|x_n - x_q\|^2 + 2\langle x_n - x_q, x_q - x^* \rangle + \|x_q - x^*\|^2 \\ &\leq 2\left(\|x^* - x_q\|^2 + \langle x_n - x_q, x_q - x^* \rangle\right) \rightarrow 0. \end{aligned} \quad (3.17)$$

That is, $\{x_n\}$ converges to $P_F(x_q)$.

This completes the proof. \square

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